

The minimality of the map $\frac{x}{\|x\|}$ for weighted energy.

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abstract

In this paper, we investigate the minimality of the map $\frac{x}{\|x\|}$ from the euclidean unit ball \mathbf{B}^n to its boundary \mathbb{S}^{n-1} for weighted energy functionals of the type $E_{p,f} = \int_{\mathbf{B}^n} f(r) \|\nabla u\|^p dx$, where f is a non-negative function. We prove that in each of the two following cases :

i) $p = 1$ and f is non-decreasing,

ii) p is an integer, $p \leq n - 1$ and $f = r^\alpha$ with $\alpha \geq 0$,

the map $\frac{x}{\|x\|}$ minimizes $E_{p,f}$ among the maps in $W^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1})$ which coincide with $\frac{x}{\|x\|}$ on $\partial\mathbf{B}^n$. We also study the case where $f(r) = r^\alpha$ with $-n+2 < \alpha < 0$ and prove that $\frac{x}{\|x\|}$ does not minimize $E_{p,f}$ for α close to $-n + 2$ and when $n \geq 6$, for α close to $4 - n$.

Keys Words : minimizing map, p -harmonic map, p -energy, weighted energy.

0.1 Introduction and statement of results

For $n \geq 3$, the map $u_0(x) = \frac{x}{\|x\|} : \mathbf{B}^n \longrightarrow \mathbb{S}^{n-1}$ from the unit ball \mathbf{B}^n of \mathbb{R}^n to its boundary \mathbb{S}^{n-1} plays a crucial role in the study of certain natural energy functionals. In particular, since the works of Hildebrandt, Kaul and Widman ([13]), this map is considered as a natural candidate to realize, for each real number $p \in [1, n)$ the minimum of the p -energy functional,

$$E_p(u) = \int_{\mathbf{B}^n} \|\nabla u\|^p dx$$

among the maps $u \in W^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1}) = \{u \in W^{1,p}(\mathbf{B}^n, \mathbb{R}^n; \|u\| = 1 \text{ a.e.}\}$ satisfying $u(x) = x$ on \mathbb{S}^{n-1} .

This question was first treated in the case $p = 2$. Indeed, the minimality of u_0 for E_2 was established by Jäger and Kaul ([16]) in dimension $n \geq 7$ and by Brezis, Coron and Lieb in dimension 3 ([2]). In [5], Coron and Gulliver proved the minimality of u_0 for E_p for any integer $p \in \{1, \dots, n-1\}$ and any dimension $n \geq 3$.

Lin ([17]) has introduced the use of the elegant null Lagrangian method (or calibration method) in this topic. Avellaneda and Lin showed the efficiency of this method in [1] where they give a simpler alternative proof to the Coron-Gulliver result. Note that several results concerning the minimizing properties of p -harmonic diffeomorphisms were also obtained in this way in particular by Coron, Helein and El Soufi, Sandier ([4], [12], [7] and [6]).

The case of non-integer p seemed to be rather difficult. It is only ten years after the Coron-Gulliver article [5], that Hardt, Lin and Wang ([10]) succeeded to prove that, for all $n \geq 3$, the map u_0 minimizes E_p for $p \in [n-1, n)$. Their proof is based on a deep studies of singularities of harmonic and minimizing maps made in the last two decades. In dimension $n \geq 7$, Wang ([20]) and Hong ([14]) have independently proved the minimality of u_0 for any $p \geq 2$ satisfying $p + 2\sqrt{p} \leq n - 2$.

In [15], Hong remarked that the minimality of the p -energy E_p , $p \in (2, n-1]$, is related to the minimization of the following weighted 2-energy :

$$\tilde{E}_p(u) = \int_{\mathbf{B}^n} r^{2-p} \|\nabla u\|^2 dx$$

where $r = \|x\|$. Indeed, using Hölder inequality, it is easy to see that if the map u_0 minimizes \tilde{E}_p , then it also minimizes E_p (see [15], p.465). Unfortunately, as we will see in Corollary 1.1 below, for many values of $p \in (2, n)$, the map u_0 is not a minimizer of \tilde{E}_p . Therefore, Theorem 6 of ([15]), asserting that u_0 minimizes \tilde{E}_p seems to be not correct and the question of whether u_0 is a minimizing map of the p -energy E_p for non-integer $p \in (2, n-1)$ is still open ¹

The aim of this paper is to study the minimizing properties of the map

¹We suspect a problem in Theorem 6 p.464 of [15]. Indeed the author claims that the quantity $G_{\varphi_1^0, \dots, \varphi_{n-1}^0}(v, p)$, which represents a weighted energy of the map v on the 3-dimensional cone \mathcal{C}_0 in \mathbf{B}^n , is uniformly proportional to the weighted energy on the euclidian ball \mathbf{B}^3 . There is no reason for this fact to be true, the orthogonal projection of \mathcal{C}_0 on to \mathbf{B}^n being not homothetic.

u_0 in regard to some weighted energy functionals of the form :

$$E_{p,f}(u) = \int_{\mathbf{B}^n} f(r) \|\nabla u\|^p dx,$$

where $p \in \{1, \dots, n-1\}$ and $f: [0, 1] \rightarrow \mathbb{R}$ is a non-negative non-decreasing continuous function. For $p = 1$, the map u_0 minimizes $E_{1,f}$ for a large class of weights. Indeed, we have the following

Theorem 0.1 *Suppose that f is a non-negative differentiable non-decreasing function. Then the map $u_0 = \frac{x}{\|x\|}$ is a minimizer of the energy $E_{1,f}$, that is, for any u in $W^{1,1}(\mathbf{B}^n, \mathbb{S}^{n-1})$ with $u(x) = x$ on \mathbb{S}^{n-1} , we have*

$$\int_{\mathbf{B}^n} f(r) \|\nabla u_0\| dx \leq \int_{\mathbf{B}^n} f(r) \|\nabla u\| dx,$$

Moreover, if f has no critical points in $(0, 1)$, then the map $u_0 = \frac{x}{\|x\|}$ is the unique minimizer of the energy $E_{1,f}$, that is, the equality in the last inequality holds if and only if $u = u_0$.

For $p \geq 2$, we restrict ourselves to power functions $f(r) = r^\alpha$,

Theorem 0.2 *For any $\alpha \geq 0$ and any integer $p \in \{1, \dots, n-1\}$, the map $u_0 = \frac{x}{\|x\|}$ is a minimizer of the energy E_{p,r^α} that is, for any u in $W^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1})$ with $u(x) = x$ on \mathbb{S}^{n-1} , we have,*

$$\int_{\mathbf{B}^n} r^\alpha \|\nabla u_0\|^p dx \leq \int_{\mathbf{B}^n} r^\alpha \|\nabla u\|^p dx.$$

Moreover, if $\alpha > 0$, then the map $u_0 = \frac{x}{\|x\|}$ is the unique minimizer of the energy E_{p,r^α} , that is the equality in the last inequality holds if and only if $u = u_0$.

The proof of these two theorems is given in section 2. It is based on a construction of an adapted null-Lagrangian. The case of $p = 1$ can be obtained passing through more direct ways and will be treated independently.

The case of weights of the form $f(r) = r^\alpha$, with $\alpha < 0$, is treated in section 3. The weighted energy $\int_{\mathbf{B}^n} r^\alpha \|\nabla u_0\|^2 dx$ of $u_0 = \frac{x}{\|x\|}$ is finite for $\alpha > -n + 2$. Hence we consider the family of maps,

$$u_a(x) = a + \lambda_a(x)(x - a), \quad a \in \mathbf{B}^n,$$

where $\lambda_a(x) \in \mathbb{R}$ is chosen such that $u_a(x) \in \mathbb{S}^{n-1}$ (that is $u_a(x)$ is the intersection point of \mathbb{S}^{n-1} with the half-line of origin a passing by x).

We study the energy $E_{2,r^\alpha}(u_a)$ of these maps and deduce the following theorem.

Theorem 0.3 Suppose that $n \geq 3$.

(i) For any $a \in \mathbf{B}^n, a \neq 0$, there exists a negative real number $\alpha_0 \in (-n+2, 0)$, such that, for any $\alpha \in (-n+2, \alpha_0]$ we have

$$\int_{\mathbf{B}^n} r^\alpha \|\nabla u_0\|^2 dx > \int_{\mathbf{B}^n} r^\alpha \|\nabla u_a\|^2 dx \quad .$$

(ii) For any integer $n \geq 6$, there exists $\alpha_0 \in (4-n, 5-n)$ such that, for any $\alpha \in (4-n, \alpha_0)$, there exists $a \in \mathbf{B}^n$ such that,

$$\int_{\mathbf{B}^n} r^\alpha \|\nabla u_0\|^2 dx > \int_{\mathbf{B}^n} r^\alpha \|\nabla u_a\|^2 dx \quad .$$

Replacing in Theorem 0.3 α by $2-p$, $p \in (2, n)$, we obtain the following corollary :

Corollary 0.1 For any $n \geq 6$, there exists $p_0 \in (n-3, n-2)$ such that, for any $p \in (p_0, n-2)$ the map $u_0 = \frac{x}{\|x\|}$ does not minimize the functional $\int_{\mathbf{B}^n} r^{2-p} \|\nabla u\|^2 dx$ among the maps $u \in W^{1,2}(\mathbf{B}^n, \mathbb{S}^{n-1})$ satisfying $u(x) = x$ on \mathbb{S}^{n-1} .

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0.2 Proof of theorems 0.1 and 0.2

Consider an integer $p \in \{1, \dots, n-1\}$ and f a differentiable, non-negative, increasing, and non-identically zero map. We can suppose without loss of generality, that $f(1) = 1$.

For any subset $I = \{i_1, \dots, i_p\} \subset \{1, \dots, n-1\}$ with $i_1 < i_2 \dots < i_p$ and for any map,

$$u = (u_1, \dots, u_n) : \mathbf{B}^n \longrightarrow \mathbb{S}^{n-1} \quad \text{in } \mathcal{C}^\infty(\mathbf{B}^n, \mathbb{S}^{n-1}) \quad \text{with } u(x) = x \text{ on } \mathbb{S}^{n-1},$$

we consider the n -form :

$$\omega_I(u) = dx_1 \wedge \dots \wedge d(f(r)u_{i_1}) \wedge \dots \wedge d(f(r)u_{i_k}) \wedge \dots \wedge dx_n$$

Lemma 0.1 We have the identity :

$$\int_{\mathbf{B}^n} \omega_I(u) = \int_{\mathbf{B}^n} \omega_I(Id) \quad \forall x \in \mathbf{B}^n \quad \text{where } Id(x) = x.$$

Proof By Stokes theorem, we have :

$$\begin{aligned}
\int_{\mathbf{B}^n} \omega_I(u) &= \int_{\mathbf{B}^n} dx_1 \wedge \cdots \wedge d(f(r)u_{i_1}) \wedge \cdots \wedge d(f(r)u_{i_p}) \wedge \cdots \wedge dx_n \\
&= \int_{\mathbf{B}^n} (-1)^{i_1-1} d\left(f(r)u_{i_1} dx_1 \wedge \cdots \wedge d(\widehat{f(r)u_{i_1}}) \wedge \right. \\
&\quad \left. \cdots \wedge d(f(r)u_{i_p}) \wedge \cdots \wedge dx_n\right) \\
&= \int_{\mathbb{S}^{n-1}} (-1)^{i_1-1} x_{i_1} dx_1 \wedge \cdots \wedge d(\widehat{f(r)u_{i_1}}) \wedge \\
&\quad \cdots \wedge d(f(r)u_{i_p}) \wedge \cdots \wedge dx_n.
\end{aligned}$$

Indeed, on \mathbb{S}^{n-1} , we have $f(r)u_{i_1} = x_{i_1}$ ($r = 1, f(1) = 1$ and $u(x) = x$). Iterating, we get the designed identities. Consider the n-form :

$$S(u) = \sum_{|I|=p} w_I(u)$$

By Lemma 0.1, we have :

$$\int_{\mathbf{B}^n} S(u) = \sum_{|I|=p} \int_{\mathbf{B}^n} w_I(u) = \sum_{|I|=p} \int_{\mathbf{B}^n} dx = C_n^p \frac{|\mathbb{S}^{n-1}|}{n},$$

where $|\mathbb{S}^{n-1}|$ is the Lebesgue measure of the sphere.

Lemma 0.2 *The n-form $S(u)$ is $O(n)$ -equivariant, that is, for any rotation R in $O(n)$, we have :*

$$S({}^tRuR)({}^tRx) = S(u)(x) \quad \forall x \in \mathbf{B}^n.$$

Proof Consider $S(u)(x)(e_1, \dots, e_n)$ where (e_1, \dots, e_n) is the standart basis of \mathbb{R}^n and notice that it is equal to $(-1)^n$ times the $(p+1)^{th}$ coefficient of the polynomial $P(\lambda) = \det(Jac(fu)(x) - \lambda Id)$ which does not change when we replace fu by tRfuR .

For any $x \in \mathbf{B}^n$, let $R \in O(n)$ be such that ${}^tRu(x) = e_n = (0, \dots, 0, 1)$. Consider $y = {}^tRx$, $v = {}^tRuR$, so that :

$$v(y) = e_n, \quad d({}^tRuR)(y)(\mathbb{R}^n) \subset e_n^\perp \quad \text{that is} \quad \frac{\partial v_n}{\partial x_j}(y) = 0 \quad \forall j \in \{1, \dots, n\}.$$

Lemma 0.3 *Let a_1, \dots, a_n be n non-negative numbers, and $p \in \{1, \dots, n-1\}$. Then :*

$$\sum_{i_1 < \dots < i_p} a_{i_1} \cdots a_{i_p} \leq \frac{1}{(n-1)^p} C_{n-1}^p \left(\sum_{j=1}^{n-1} a_j \right)^p.$$

Proof See for instance Hardy coll.[4], theorem 52.

Let $I = \{i_1, \dots, i_p\} \subset \{1, \dots, n\}$. We have :
if $i_p \neq n$,

$$\begin{aligned} \omega_I(v)(y) &= (dx_1 \wedge \cdots \wedge d(f(r)v_{i_1}) \wedge \cdots \wedge d(f(r)v_{i_k}) \wedge \cdots \wedge dx_n)(y) \\ &= |f(r)|^p (dx_1 \wedge \cdots \wedge dv_{i_1} \wedge \cdots \wedge dv_{i_k} \wedge \cdots \wedge dx_n)(y). \end{aligned}$$

Indeed, $\forall j \leq n-1$, $d(f(r)v_j(y)) = d(f(r))v_j(y) + f(r)dv_j(y) = f(r)dv_j(y)$
since $v(y) = e_n$.

If $i_p = n$,

$$\omega_I(v)(y) = |f(r)|^{p-1} (dx_1 \wedge \cdots \wedge dv_{i_1} \wedge \cdots \wedge df)(y).$$

Indeed, $d(f(r)v_n)(y) = df(y)v_n(y) + f(r)dv_n(y) = df(y)$ (as $dv(y) \subset e_n^\perp$). The Hadamard inequality gives :

$$\begin{aligned} |S(v)(y)| &= \left| \sum_{|I|=p} \omega_I(v)(y) \right| \leq |f(r)|^p \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq n-1} \|dx_1\| \cdots \|dv_{i_1}\| \\ &\quad \cdots \|dv_{i_p}\| \cdots \|dx_n\|(y) \\ &+ |f(r)|^{p-1} \sum_{1 \leq i_1 < i_2 < \dots < i_{p-1} \leq n-1} \|dx_1\| \cdots \|dv_{i_1}\| \\ &\quad \cdots \|dv_{i_p}\| \cdots \|df\|(y) \\ &\leq |f(r)|^p \left(\sum_{1 \leq i_1 < i_2 < \dots < i_p \leq n-1} \|dx_1\|^2 \cdots \|dv_{i_1}\|^2 \cdots \right. \\ &\quad \left. \cdots \|dv_{i_p}\|^2 \cdots \|dx_n\|^2(y) \right)^{\frac{1}{2}} (C_n^p)^{\frac{1}{2}} \\ &+ f'(r)f(r)^{p-1} \sum_{1 \leq i_1 < i_2 < \dots < i_{p-1} \leq n-1} \|dx_1\| \cdots \|dv_{i_1}\| \\ &\quad \cdots \|dv_{i_p}\|(y). \end{aligned}$$

The Hardy inequality gives, after integration and using the fact that $\|\nabla u\| = \|\nabla v\|$,

$$\begin{aligned} \frac{C_n^p}{n} |\mathbb{S}^{n-1}| &\leq \frac{C_{n-1}^p}{(n-1)^{p/2}} \int_{\mathbf{B}^n} f^p(r) \|\nabla u\|^p dx \\ &+ \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \int_{\mathbf{B}^n} f'(r) f^{p-1}(r) \|\nabla u\|^{p-1} dx. \quad (1) \end{aligned}$$

Remark : If f' is positive and if equality holds in (1), then, $\forall i \leq n-1$, $y_i = 0$ and $y_n = \pm \frac{x}{\|x\|}$, which implies that $u(x) = \pm \frac{x}{\|x\|}$.

Proof of the Theorem 1.1 Inequality (1) give

$$|\mathbb{S}^{n-1}| \leq \sqrt{n-1} \int_{\mathbf{B}^n} f(r) \|\nabla u\| dx + \int_{\mathbf{B}^n} f'(r) dx.$$

Hence :

$$\begin{aligned} \int_{\mathbf{B}^n} f \|\nabla u\| dx &\geq \frac{|\mathbb{S}^{n-1}|}{\sqrt{n-1}} \left(1 - \int_0^1 f'(r) r^{n-1} dr\right) \\ \int_{\mathbf{B}^n} f \|\nabla u\| dx &\geq \sqrt{n-1} |\mathbb{S}^{n-1}| \int_0^1 f(r) r^{n-2} dr = \int_{\mathbf{B}^n} f(r) \|\nabla u_0\| dx. \end{aligned}$$

To see the uniqueness it suffices to refer to the remark above. It gives that for any $x \in \mathbf{B}^n$, $u(x) = \frac{x}{\|x\|}$ or $u(x) = -\frac{x}{\|x\|}$. As $u(x) = x$ on the unit sphere, we have, for any $x \in \mathbf{B}^n \setminus \{0\}$, $u(x) = \frac{x}{\|x\|}$. ■

Proof of the Theorem 1.2. Let α be a positive real number. From inequality (1) we have :

$$\frac{C_n^p}{n} |\mathbb{S}^{n-1}| \leq \frac{C_{n-1}^p}{(n-1)^{p/2}} \int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u\|^p dx + \alpha \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \int_{\mathbf{B}^n} r^{\alpha p-1} \|\nabla u\|^{p-1} dx.$$

By Hölder inequality, we have, setting $q = \frac{p}{p-1}$:

$$\begin{aligned} \frac{C_n^p}{n} |\mathbb{S}^{n-1}| &\leq \frac{C_{n-1}^p}{(n-1)^{p/2}} \int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u\|^p dx \\ &+ \alpha \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \left(\int_{\mathbf{B}^n} r^{p(\alpha-1)} dx \right)^{1/p} \left(\int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u\|^p dx \right)^{1/q} \\ &\leq \frac{C_{n-1}^p}{(n-1)^{p/2}} \int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u\|^p dx \\ &+ \alpha \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \frac{|\mathbb{S}^{n-1}|^{1/p}}{(n+p(\alpha-1))^{1/p}} \left(\int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u\|^p dx \right)^{1/q} \end{aligned}$$

Consider the polynomial function :

$$P(t) = \frac{C_{n-1}^p}{(n-1)^{p/2}} t^q + \alpha \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \frac{|\mathbb{S}^{n-1}|^{1/p}}{(n+p(\alpha-1))^{1/p}} t - \frac{C_n^p}{n} |\mathbb{S}^{n-1}|.$$

Setting $A = (\int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u\|^p)^{1/q}$ and $B = (\int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u_0\|^p)^{1/q}$, we get $P(A) \geq 0$ while

$$\begin{aligned} P(B) &= \frac{C_{n-1}^{p-1}}{n+p(\alpha-1)} |\mathbb{S}^{n-1}| + \alpha \frac{C_{n-1}^{p-1}}{n+p(\alpha-1)} |\mathbb{S}^{n-1}| - \frac{C_n^p}{n} |\mathbb{S}^{n-1}| \\ &= \frac{C_{n-1}^{p-1}}{n+p(\alpha-1)} |\mathbb{S}^{n-1}| \left(\frac{n-p}{n} + \alpha - \frac{C_n^p}{nC_{n-1}^{p-1}} (n+p(\alpha-1)) \right) \\ &= 0. \end{aligned}$$

On the other hand, $\forall t \geq 0$, $P'(t) > 0$. Hence, P is increasing in $[0, +\infty)$ and is equal to zero only for B . Necessarily, we have $A \geq B$.

Moreover, if $\alpha > 0$, $A = B$ implies that equality in the inequality (1) holds. Referring to the remark above, and as $u_0(x) = x$ on the sphere, we have $u = u_0 = \frac{x}{\|x\|}$. Replacing α by α/p we finish the prove of the theorem. \blacksquare

0.3 The energy of a natural family of maps.

Let $a = (\theta, \dots, 0)$ be a point of \mathbf{B}^n with $0 < \theta < 1$ and consider the map,

$$u_a(x) = a + \lambda_a(x)(x - a),$$

where $\lambda_a(x) > 0$ is chosen so that $u_a(x) \in \mathbb{S}^{n-1}$ for any $x \in \mathbf{B}^n \setminus \{0\}$,

$$\lambda_a(x) = \frac{\sqrt{\Delta_a(x)} - (a|x - a)}{\|x - a\|^2}$$

and

$$\Delta_a(x) = (1 - \|a\|^2)\|x - a\|^2 + (a|x - a)^2.$$

Notice that $u_a(x) = x$ as soon as x is on the sphere. If we denote by $\{e_i\}_{i \in \{1, \dots, n\}}$ the standard basis of \mathbb{R}^n , then, $\forall i \leq n$, we have,

$$\begin{aligned}
\|du_a(x).e_i\|^2 &= \left(\frac{\sqrt{\Delta_a} - (a|x-a)}{\|x-a\|^2} \right)^2 \\
&+ \left[-2 \frac{(x-a|e_i)}{\|x-a\|^4} \left(\sqrt{\Delta_a} - (a|x-a) \right) \right. \\
&+ \frac{(1-\|a\|^2)(x-a|e_i) + (x-a|a)(a|e_i)}{\sqrt{\Delta_a}\|x-a\|^2} \\
&\left. - \frac{(a|e_i)}{\|x-a\|^2} \right]^2 \|x-a\|^2 \\
&+ 2 \left(\frac{\sqrt{\Delta_a} - (a|x-a)}{\|x-a\|^2} \right) \left(-2 \frac{(x-a|e_i)}{\|x-a\|^4} \left(\sqrt{\Delta_a} - (a|x-a) \right) \right. \\
&+ \frac{(1-\|a\|^2)(x-a|e_i) + (x-a|a)(a|e_i)}{\sqrt{\Delta_a}\|x-a\|^2} \\
&\left. - \frac{(a|e_i)}{\|x-a\|^2} \right) (x-a|e_i).
\end{aligned}$$

Let us prove that, for each $\alpha \in (-n, 0)$, $\int_{\mathbf{B}^n} r^\alpha \|\nabla u_a\| dx$ is finite. Consider the map :

$$\begin{aligned}
F : \mathbb{R}^+ \times \mathbb{S}^{n-1} &\longrightarrow \mathbb{R}^n \\
(r, s) &\longmapsto a + rs = x.
\end{aligned}$$

Then, we have,

$$F^*(\|\nabla u_a\|^2 dx) = \frac{1}{r^2} \sum_{i=1}^n H_{i,a}(s) r^{n-1} dr \wedge ds,$$

where $H_{i,a}(s)$ is given on the sphere by,

$$\begin{aligned}
H_{i,a}(s) &= \left((1 - \|a\|^2 + (a|s)^2)^{1/2} - (a|s) \right)^2 \\
&+ \left[-2(s|e_i) \left((1 - \|a\|^2 + (a|s)^2)^{1/2} - (s|a) \right) \right. \\
&+ \left. \frac{(1 - \|a\|^2)(s|e_i) + (a|e_i)(s|a)}{(1 - \|a\|^2 + (a|s)^2)^{1/2}} - (a|e_i) \right]^2 \\
&+ 2 \left((1 - \|a\|^2 + (a|s)^2)^{1/2} - (a|s) \right) \\
&\left(-2(s|e_i) \left((1 - \|a\|^2 + (a|s)^2)^{1/2} - (s|a) \right) \right. \\
&+ \left. \frac{(1 - \|a\|^2)(s|e_i) + (a|e_i)(s|a)}{(1 - \|a\|^2 + (a|s)^2)^{1/2}} - (a|e_i) \right) (s|e_i).
\end{aligned}$$

It is clear that $H_{i,a}(s)$ is continuous on \mathbb{S}^{n-1} . Therefore, near the point a , as $n \geq 3$, the map $\|x\|^\alpha \|\nabla u_a\|$ is integrable. Furthermore, near the point 0, as $\alpha > -n$, this map is also integrable. In conclusion, for any $\alpha \in (-n, 0)$, the energy $E_{r^\alpha, 2}(u_a)$ is finite.

Proof of Theorem 1.3(i). Since we have

$$E_{2,r^\alpha}(u_0) = \int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u_0\|^2 dx = \frac{|\mathbb{S}^{n-1}|(n-1)}{n+\alpha-2},$$

the energy $E_{2,r^\alpha}(u_0)$ goes to infinity as $\alpha \rightarrow -n+2$. On the other hand, as the energy $E_{2,r^\alpha}(u_a)$ is continuous in α , there exists a real number $\alpha_0 \in (-n+2, 0)$ such that, $\forall \alpha, 2-n < \alpha \leq \alpha_0$,

$$\int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u_0\|^2 dx > \int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u_a\|^2 dx.$$

Proof of Theorem 1.3(ii). Since $a = (\theta, 0, \dots, 0)$, we will study the function,

$$G(\theta) = E_{2,r^\alpha}(u_a) = \int_{\mathbf{B}^n} r^\alpha \|\nabla u_a\|^2 dx.$$

Precisely, we will show that for any $\alpha \in (5-n, 4-n)$, G is two times differentiable at $\theta = 0$ with $\frac{dG}{d\theta}(0) = 0$ and, when α is sufficiently close to $4-n$, $\frac{d^2G}{d\theta^2}(0) < 0$. Assertion (ii) of Theorem 1.3 then follows immediately.

We have,

$$\begin{aligned}
H_{i,a}(s) = H_{i,\theta}(s) &= \left(\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1 \right)^2 \\
&+ \left(-2s_i \left(\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1 \right) + \frac{(1-\theta^2)s_i + \delta_{i1}\theta^2 s_1}{\sqrt{1-\theta^2+\theta^2 s_1^2}} - \delta_{i1}\theta \right)^2 \\
&+ 2 \left(\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1 \right) \left(-2s_i \left(\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1 \right) \right. \\
&\quad \left. + \frac{(1-\theta^2)s_i + \delta_{i1}\theta^2 s_1}{\sqrt{1-\theta^2+\theta^2 s_1^2}} - \delta_{i1}\theta \right) s_i,
\end{aligned}$$

where $\delta_{ij} = 0$ if $i \neq j$ and 1 else.

We notice that $H_{i,\theta}(s)$ is bounded on $[0, 1] \times \mathbb{S}^{n-1}$. Indeed, for all $x, y, z \in [0, 1]$, excepting $(x, y) = (0, 1)$, we have,

$$\left| \frac{x}{\sqrt{1-y^2+y^2x^2}} \right| \leq 1 \quad \text{and} \quad \left| \frac{(1-y^2)z}{\sqrt{1-y^2+y^2x^2}} \right| \leq 1.$$

Then, for almost all $(s, \theta) \in \mathbb{S}^{n-1} \times [0, 1]$, we have,

$$\left| \frac{(1-\theta^2)s_i + \delta_{i1}\theta^2 s_1}{\sqrt{1-\theta^2+\theta^2 s_1^2}} \right| \leq 1,$$

and the others terms are continuous in $[0, 1] \times \mathbb{S}^{n-1}$.

We have,

$$\begin{aligned}
E_{2,r^\alpha}(u_a) &= \int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u_a\|^2 dx = \int_{\mathbf{B}^n} \|a + rs\|^\alpha r^{n-3} H(\theta, s) dr ds \\
&= \int_{\mathbf{S}^{n-1}} H(\theta, s) \left(\int_0^{\gamma_\theta(s)} ((r + \theta s_1)^2 + \theta^2(1 - s_1^2))^{\alpha/2} r^{n-3} dr \right) ds,
\end{aligned}$$

where $\gamma_\theta(s) = \sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1$ and $H(\theta, s) = \sum_{i=1}^n H_{i,\theta}(s)$. We notice that $H(\theta, s)$ is indefinitely differentiable in $(-1/2, 1/2) \times \mathbb{S}^{n-1}$. Let C_n be a positive real number so that, $\forall (\theta, s) \in (-1/2, 1/2) \times \mathbb{S}^{n-1}$

$$|H(\theta, s)| \leq C_n, \quad \left| \frac{\partial H(\theta, s)}{\partial \theta} \right| \leq C_n, \quad \left| \frac{\partial^2 H(\theta, s)}{\partial \theta^2} \right| \leq C_n.$$

Furthermore, we have,

$$H(\theta, s) = (n-1) - 2(n-1)s_1\theta + ((2n-3)s_1^2 - n + 2)\theta^2 + o(\theta^2). \quad (A)$$

Let us set $\rho = r + \theta s_1$, $\beta(\theta, s) = \sqrt{1 - \theta^2 + \theta^2 s_1^2}$ and

$$F(\theta, s) = \int_{\theta s_1}^{\beta(\theta, s)} (\rho - \theta s_1)^{n-3} (\rho^2 + \theta^2(1 - s_1^2))^{\alpha/2} d\rho.$$

Notice that $\rho \in [-1, 3]$. Then, $G(\theta) = \int_{\mathbb{S}^{n-1}} H(\theta, s) F(\theta, s) ds$. Let us set $g(\rho, \theta, s) = (\rho - \theta s_1)^{n-3} (\rho^2 + \theta^2(1 - s_1^2))^{\alpha/2}$.

Lemma 0.4 *The map $\theta \mapsto G(\theta)$ is continuous on $(-1/2, 1/2)$ and continuously differentiable on $(-1/2, 1/2) \setminus \{0\}$ for any $\alpha > 3 - n$.*

Proof We have, $\forall s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$,

$$\frac{(\rho - \theta s_1)^2}{(\rho^2 + \theta^2(1 - s_1^2))} \leq \frac{2}{1 - s_1^2} \quad (1.1)$$

Indeed, $(1 - s_1^2)(\rho - \theta s_1)^2 \leq 2(1 - s_1^2)(\rho^2 + \theta^2) \leq 2(\rho^2 + \theta^2(1 - s_1^2))$. And then,

$$g(\rho, \theta, s) \leq \frac{2^{\frac{n-3}{2}}}{(1 - s_1^2)^{\frac{n-3}{2}}} (\rho^2 + \theta^2(1 - s_1^2))^{\frac{\alpha+n-3}{2}}. \quad (1.2)$$

Since $\alpha > 3 - n$ we deduce that the map $(\rho, \theta) \rightarrow g(\rho, \theta, s)$ is continuous on $(-1/2, 1/2) \times [-1, 3]$. Hence, the map $z \mapsto \int_0^z g(\rho, \theta, s) d\rho$ is differentiable on $[-1, 3]$ and,

$$\frac{\partial}{\partial z} \int_0^z g(\rho, \theta, s) d\rho = g(z, \theta, s).$$

Furthermore, for any $\rho \in [-1, 3]$, the map $\theta \mapsto g(\rho, \theta, s)$ is differentiable and

$$\begin{aligned} \frac{\partial g}{\partial \theta}(\rho, \theta, s) &= -(n-3)s_1(\rho - \theta s_1)^{n-4}(\rho^2 + \theta^2(1 - s_1^2))^{\frac{\alpha}{2}} \\ &+ \frac{\alpha}{2}(\rho - \theta s_1)^{n-3}2\theta(1 - s_1^2)(\rho^2 + \theta^2(1 - s_1^2))^{\frac{\alpha}{2}-1}. \end{aligned}$$

Let a, b be two real in $(0, 1/2)$ with $a < b$. We have for any $|\theta| \in (a, b)$, for any $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$,

$$\begin{aligned} \left| \frac{\partial g}{\partial \theta}(\rho, \theta, s) \right| &\leq (n-3)4^{n-4}(a^2(1 - s_1^2))^{\frac{\alpha}{2}} \\ &+ |\alpha|4^{n-3}(1 - s_1^2)(a^2(1 - s_1^2))^{\frac{\alpha}{2}-1}. \end{aligned} \quad (1.3)$$

This shows that $\theta \mapsto \int_0^z g(\rho, \theta, s) d\rho$ is differentiable on $(-1/2, 1/2) \setminus \{0\}$ and

$$\frac{\partial}{\partial \theta} \int_0^z g(\rho, \theta, s) d\rho = \int_0^z \frac{\partial g}{\partial \theta}(\rho, \theta, s) d\rho.$$

Moreover the map $(z, \theta) \mapsto \int_0^z \frac{\partial g}{\partial \theta}(\rho, \theta, s) d\rho$ is continuous in $[-1, 3] \times (-1/2, 1/2) \setminus \{0\}$. Indeed, $\theta \mapsto \frac{\partial g}{\partial \theta}(\rho, \theta, s)$ is clearly continuous on $(-1/2, 1/2) \setminus \{0\}$ and from (1.3) and by Lebesgue Theorem, $\theta \mapsto \int_0^z \frac{\partial g}{\partial \theta}(\rho, \theta, s) d\rho$ is continuous on $(-1/2, 1/2) \setminus \{0\}$. Then, for any $\epsilon > 0$, we will have for any sufficiently small h, k ,

$$\begin{aligned} \left| \int_0^{z+h} \frac{\partial g}{\partial \theta}(\rho, \theta + k, s) d\rho - \int_0^z \frac{\partial g}{\partial \theta}(\rho, \theta, s) d\rho \right| &\leq \left| \int_0^z \frac{\partial g}{\partial \theta}(\rho, \theta + k, s) d\rho \right. \\ &\quad \left. - \int_0^z \frac{\partial g}{\partial \theta}(\rho, \theta, s) d\rho \right| \\ &\quad + \left| \int_z^{z+h} \frac{\partial g}{\partial \theta}(\rho, \theta + k, s) d\rho \right| \\ &\leq \epsilon. \end{aligned}$$

The map $(z, \theta) \mapsto \int_0^z g(\rho, \theta, s) d\rho$ is differentiable on $[-1, 3] \times (-1/2, 1/2) \setminus \{0\}$ and the map $\theta \mapsto F(\theta, s)$ is differentiable in $(-1/2, 1/2) \setminus \{0\}$ and for any $\theta \in (-1/2, 1/2) \setminus \{0\}$,

$$\begin{aligned} \frac{\partial F}{\partial \theta}(\theta, s) &= \frac{\partial \beta}{\partial \theta}(\theta, s) g(\beta(\theta, s), \theta, s) - s_1 g(\theta s_1, \theta, s) + \int_{\theta s_1}^{\beta(\theta, s)} \frac{\partial g}{\partial \theta}(\rho, \theta, s) d\rho \\ &= \frac{\theta(s_1^2 - 1)}{(1 - \theta^2 + \theta^2 s_1^2)^{1/2}} ((1 - \theta^2 + \theta^2 s_1^2)^{1/2} - \theta s_1)^{n-3} \\ &\quad + \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho + \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho, \end{aligned}$$

where,

$$g_1(\rho, \theta, s) = -(n-3)s_1(\rho - \theta s_1)^{n-4}(\rho^2 + \theta^2(1 - s_1^2))^{\frac{\alpha}{2}}$$

and

$$g_2(\rho, \theta, s) = \frac{\alpha}{2}(\rho - \theta s_1)^{n-3} 2\theta(1 - s_1^2)(\rho^2 + \theta^2(1 - s_1^2))^{\frac{\alpha}{2}-1}.$$

Now, the map $\theta \mapsto F(\theta, s)$ is continuous on $(-1/2, 1/2)$. Indeed, since the map $\theta \mapsto g(\rho, \theta, s) d\rho$ is continuous on $(-1/2, 1/2)$ and from (1.2) $\theta \mapsto \int_0^z g(\rho, \theta, s) d\rho$ is continuous on $(-1/2, 1/2)$. Then, for any $\epsilon > 0$, we have $\forall h, k$ sufficiently small,

$$\begin{aligned} \left| \int_0^{z+h} g(\rho, \theta + k, s) d\rho - \int_0^z g(\rho, \theta, s) d\rho \right| &\leq \left| \int_0^z g(\rho, \theta + k, s) d\rho \right. \\ &\quad \left. - \int_0^z g(\rho, \theta, s) d\rho \right| \\ &\quad + \left| \int_z^{z+h} g(\rho, \theta + k, s) d\rho \right| \\ &\leq \epsilon. \end{aligned}$$

Then, the map $(z, \theta) \mapsto \int_0^z g(\rho, \theta, s) d\rho$ is continuous on $[-1, 3] \times (-1/2, 1/2)$ and consequently $\theta \mapsto F(\theta, s)$ is continuous on $(-1/2, 1/2)$.

Now, we know that $\theta \mapsto H(\theta, s)F(\theta, s)$ is continuous on $(-1/2, 1/2)$ and differentiable on $(-1/2, 1/2) \setminus \{0\}$. Furthermore from (1.2), we have, for any $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$,

$$|H(\theta, s)F(\theta, s)| \leq 3.2^{\frac{n-3}{2}} 10^{\frac{\alpha+n-3}{2}} C_n \cdot \frac{1}{(1-s_1^2)^{\frac{n-3}{2}}}. \quad (1.4)$$

$$\left| \frac{\partial H}{\partial \theta}(\theta, s)F(\theta, s) \right| \leq 3.2^{\frac{n-3}{2}} 10^{\frac{\alpha+n-3}{2}} C_n \cdot \frac{1}{(1-s_1^2)^{\frac{n-3}{2}}}. \quad (1.5)$$

Consider the map $\eta : (\theta, s) \mapsto \eta(\theta, s) = \frac{\theta(s_1^2-1)}{\sqrt{1-\theta^2+\theta^2 s_1^2}} ((1-\theta^2+\theta^2 s_1^2)^{1/2} - \theta s_1)^{n-3}$.

This map is indefinitely differentiable on $(-1/2, 1/2) \times \mathbb{S}^{n-1}$. Let B_n be a positive real number so that, $\forall (\theta, s) \in (-1/2, 1/2) \times \mathbb{S}^{n-1}$,

$$|\eta(\theta, s)| \leq B_n \quad \left| \frac{\partial \eta}{\partial \theta}(\theta, s) \right| \leq B_n.$$

Considering $a, b \in (0, 1/2)$ with $a < b$ we have, for any $\theta \in (a, b)$, for any $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$,

$$\begin{aligned} \left| H(\theta, s) \frac{\partial F}{\partial \theta}(\theta, s) \right| &\leq \left(B_n + 3(n-3) \cdot 4^{n-4} \cdot a^\alpha (1-s_1^2)^{\frac{\alpha}{2}} \right. \\ &\quad \left. + |3\alpha| \cdot 4^{n-3} a^{\alpha-1} (1-s_1^2)^{\frac{\alpha}{2}} \right) C_n. \end{aligned} \quad (1.6)$$

Since the maps $s \mapsto \frac{1}{(1-s_1^2)^{\frac{n-3}{2}}}$ and $s \mapsto (1-s_1^2)^{\frac{\alpha}{2}}$ are integrable on \mathbb{S}^{n-1} , we deduce that $\theta \mapsto G(\theta)$ is continuous on $(-1/2, 1/2)$ and continuously differentiable on $(-1/2, 1/2) \setminus \{0\}$.

Lemma 0.5 *The map $\theta \mapsto G(\theta)$ is differentiable at 0 and $\frac{dG}{d\theta}(0) = 0$.*

Proof Since for any $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$, $\theta \mapsto F(\theta, s)$ is continuous on $(-1/2, 1/2)$ from (A) we have,

$$\frac{\partial H}{\partial \theta}(\theta, s)F(\theta, s) \xrightarrow{\theta \rightarrow 0} \frac{\partial H}{\partial \theta}(0, s)F(0, s) = -2(n-1)s_1 \int_0^1 \rho^{n-3+\alpha} d\rho = \frac{-2(n-1)s_1}{n-2+\alpha}.$$

From (1.5) and Lebesgue Theorem we have,

$$\int_{\mathbb{S}^{n-1}} \frac{\partial H}{\partial \theta}(\theta, s)F(\theta, s) ds \xrightarrow{\theta \rightarrow 0} \int_{\mathbb{S}^{n-1}} \frac{-2(n-1)s_1}{n-2+\alpha} ds = 0.$$

Moreover, it is clear that,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \eta(\theta, s) ds \xrightarrow{\theta \rightarrow 0} 0.$$

Let $J(m, n)$ be the integral,

$$J(m, n) = \int_{\frac{s_1}{\sqrt{1-s_1^2}}}^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} (\sqrt{1-s_1^2}t - s_1)^m (t^2 + 1)^n dt.$$

Notice that $J(m, n)$ converges as θ goes to 0 if and only if $m + 2n < -1$. Consider the change of variables $\rho = t\theta\sqrt{1-s_1^2}$ if $\theta > 0$. If $\theta < 0$, then we set $\rho = -t\theta\sqrt{1-s_1^2}$ and conclusion will be the same. Hence, we assume that $\theta > 0$. Then,

$$\int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho = -(n-3)s_1(1-s_1^2)^{\frac{1+\alpha}{2}} \theta^{n-3+\alpha} J(n-4, \frac{\alpha}{2}).$$

$$\int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho = \alpha \theta^{n-3+\alpha} (1-s_1^2)^{\frac{1+\alpha}{2}} J(n-3, \frac{\alpha}{2} - 1).$$

First case : $\alpha \geq 4 - n$.

$J(n-4, \frac{\alpha}{2})$ and $J(n-3, \frac{\alpha}{2} - 1)$ go to $+\infty$ as $\theta \rightarrow 0$. Furthermore, we have,

$$\begin{aligned} J(n-4, \frac{\alpha}{2}) &\underset{0}{\sim} (1-s_1^2)^{\frac{n-4}{2}} \int_{\frac{s_1}{\sqrt{1-s_1^2}}}^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} t^{n-4+\alpha} dt \\ J(n-4, \frac{\alpha}{2}) &\underset{0}{\sim} \frac{1}{n-3+\alpha} \frac{1}{\theta^{n-3+\alpha}} (1-s_1^2)^{\frac{-1-\alpha}{2}}. \end{aligned}$$

Since $t^{n+\alpha-5}$ may be equal to zero at zero, we write,

$$\begin{aligned} J(n-3, \frac{\alpha}{2} - 1) &= \int_{\frac{s_1}{\sqrt{1-s_1^2}}}^1 (\sqrt{1-s_1^2}t - s_1)^{n-3} (t^2 + 1)^{\frac{\alpha}{2}-1} dt \\ &\quad + \int_1^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} (\sqrt{1-s_1^2}t - s_1)^{n-3} (t^2 + 1)^{\frac{\alpha}{2}-1} dt. \end{aligned}$$

We have,

$$J(n-3, \frac{\alpha}{2} - 1) \underset{0}{\sim} (1-s_1^2)^{\frac{n-3}{2}} \int_1^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} t^{n-5+\alpha} dt.$$

Then, if $\alpha \neq 4 - n$,

$$J(n-3, \frac{\alpha}{2} - 1) \underset{0}{\sim} \frac{1}{n-4+\alpha} \frac{1}{\theta^{n-4+\alpha}} (1-s_1^2)^{\frac{1-\alpha}{2}},$$

and note that if $\alpha = 4 - n$, $J(n-3, \frac{\alpha}{2} - 1) \underset{0}{\sim} -(1-s_1^2)^{\frac{n-3}{2}} \ln(\theta^2(1-s_1^2))$. Hence, by (A) we have,

$$\begin{aligned} H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho &= -H(\theta, s)(n-3)s_1(1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} I_1 \\ &\xrightarrow{\theta \rightarrow 0} -\frac{(n-3)(n-1)}{n-3+\alpha} s_1, \end{aligned}$$

and

$$H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho = H(\theta, s) \alpha (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} I_2 \xrightarrow{\theta \rightarrow 0} 0.$$

Observe that $\frac{|s_1|}{\sqrt{1-s_1^2}} \leq \sqrt{\frac{1}{\theta^2(1-s_1^2)} - 1}$. Indeed, $s_1^2 \theta^2 \leq 1 - \theta^2 + \theta^2 s_1^2$. It follows from (1.1) that

$$(\rho - \theta s_1)^{n-4} \leq \frac{2^{\frac{n-4}{2}}}{(1-s_1^2)^{\frac{n-4}{2}}} (\rho^2 + \theta^2(1-s_1^2))^{\frac{n-4}{2}}.$$

Recall that $\rho = t\theta\sqrt{1-s_1^2}$. Since $\alpha \geq 4 - n$, we have, for any $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$,

$$\begin{aligned} \left| H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho \right| &\leq 2C_n(n-3) 2^{\frac{n-4}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)} - 1}} (t^2 + 1)^{\frac{n-4+\alpha}{2}} dt \\ &\leq C_n(n-3) 2^{\frac{n-2}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \sqrt{\frac{1}{\theta^2(1-s_1^2)} - 1} \left(\frac{1}{\theta^2(1-s_1^2)} \right)^{\frac{n-4+\alpha}{2}} \\ &\leq C_n(n-3) 2^{\frac{n-2}{2}} (1-s_1^2)^{\frac{-n+4}{2}} \sqrt{1 - \theta^2(1-s_1^2)} \\ &\leq C_n(n-3) 2^{\frac{n-1}{2}} (1-s_1^2)^{\frac{-n+4}{2}}. \end{aligned}$$

Since $s \mapsto (1-s_1^2)^{\frac{-n+4}{2}}$ is integrable on \mathbb{S}^{n-1} , by Lebesgue Theorem we have,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho ds \xrightarrow{\theta \rightarrow 0} - \int_{\mathbb{S}^{n-1}} \frac{(n-3)(n-1)}{n-3+\alpha} s_1 ds = 0.$$

Moreover, we have, for any $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$, since $\alpha + n - 5 \geq 0$,

$$\begin{aligned}
\left| H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho \right| &\leq 2C_n |\alpha| 2^{\frac{n-3}{2}} (1 - s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\
&\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)} - 1}} (t^2 + 1)^{\frac{n-5+\alpha}{2}} dt \\
&\leq C_n |\alpha| 2^{\frac{n-2}{2}} (1 - s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\
&\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)} - 1}} \frac{1}{(t^2 + 1)} dt \left(\frac{1}{\theta^2(1 - s_1^2)} \right)^{\frac{n-3+\alpha}{2}} \\
&\leq C_n |\alpha| 2^{\frac{n-2}{2}} \frac{\pi}{2} (1 - s_1^2)^{\frac{-n+4}{2}}.
\end{aligned}$$

Then, by Lebesgue Theorem,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho ds \xrightarrow{\theta \rightarrow 0} 0.$$

Second case : $3 - n < \alpha < 4 - n$.

For the same reasons that when $\alpha \geq 4 - n$, we have,

$$H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho \xrightarrow{\theta \rightarrow 0} -\frac{(n-3)(n-1)}{n-3+\alpha} s_1.$$

Furthermore, as $4 - n > \alpha > 3 - n$, $\forall s \in \mathbb{S}^{n-1} \setminus \{(-1, 0, \dots, 0), (1, 0, \dots, 0)\}$,

$$\begin{aligned}
\left| H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho \right| &\leq 2C_n (n-3) 2^{\frac{n-4}{2}} (1 - s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\
&\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)} - 1}} (t^2 + 1)^{\frac{n-4+\alpha}{2}} dt \\
&\leq C_n (n-3) 2^{\frac{n-2}{2}} (1 - s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\
&\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)} - 1}} (t^2)^{\frac{n-4+\alpha}{2}} dt \\
&\leq \frac{C_n (n-3) 2^{\frac{n-2}{2}} (1 - s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha}}{n-3+\alpha} \left(\frac{1}{\theta^2(1 - s_1^2)} - 1 \right)^{\frac{n-3+\alpha}{2}} \\
&\leq \frac{C_n (n-3) 2^{\frac{2n-7+\alpha}{2}} (1 - s_1^2)^{\frac{4-n}{2}}}{n-3+\alpha}.
\end{aligned}$$

Then, by Lebesgue Theorem,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho ds \xrightarrow{\theta \rightarrow 0} -\int_{\mathbb{S}^{n-1}} \frac{(n-3)(n-1)}{n-3+\alpha} s_1 ds = 0.$$

Moreover, $J(n-3, \frac{\alpha}{2}-1)$ is finite when $\theta \rightarrow 0$ then, as $\alpha > 3-n$, Furthermore,

$$H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho ds \xrightarrow{\theta \rightarrow 0} 0.$$

$$\begin{aligned} \left| H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho \right| &\leq 2C_n |\alpha| 2^{\frac{n-3}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} (t^2+1)^{\frac{n+\alpha-5}{2}} dt \\ &\leq C_n |\alpha| 2^{\frac{n-1}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} \frac{1}{(t^2+1)} dt \left(\frac{1}{\theta^2(1-s_1^2)} \right)^{\frac{n-3+\alpha}{2}} \\ &\leq C_n |\alpha| 2^{\frac{n-1}{2}} (1-s_1^2)^{\frac{-n+4}{2}} \int_0^{+\infty} \frac{1}{(t^2+1)} dt. \end{aligned}$$

Then, by Lebesgue Theorem,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho ds \xrightarrow{\theta \rightarrow 0} 0.$$

Finally, we have

$$\frac{dG}{d\theta}(\theta) \xrightarrow{\theta \rightarrow 0} 0.$$

By Lemma 1.4 we deduce that G is differentiable at 0 and $\frac{dG}{d\theta}(0) = 0$.

Lemma 0.6 *The map $\theta \rightarrow G(\theta)$ is two times differentiable on $(-1/2, 1/2) \setminus \{0\}$.*

Proof We know that the map $\theta \rightarrow \frac{\partial H}{\partial \theta}(\theta, s) F(\theta, s)$ is differentiable on $(-1/2, 1/2) \setminus \{0\}$. The maps $\theta \rightarrow \eta(\theta, s)$, $\theta \rightarrow g_1(\rho, \theta, s)$, $\theta \rightarrow g_2(\rho, \theta, s)$ are differentiable on $(-1/2, 1/2) \setminus \{0\}$. We have,

$$\begin{aligned} \frac{\partial \eta}{\partial \theta}(\theta, s) &= \frac{(s_1^2 - 1) \sqrt{1 - \theta^2 + \theta^2 s_1^2} - \theta(s_1^2 - 1) \frac{\theta(s_1^2 - 1)}{\sqrt{1 - \theta^2 + \theta^2 s_1^2}}}{1 - \theta^2 + \theta^2 s_1^2} \\ &\quad \times (\sqrt{1 - \theta^2 + \theta^2 s_1^2} - \theta s_1)^{n-3} \\ &\quad + \frac{(n-3)\theta(s_1^2 - 1)}{\sqrt{1 - \theta^2 + \theta^2 s_1^2}} \left(\frac{\theta(s_1^2 - 1)}{\sqrt{1 - \theta^2 + \theta^2 s_1^2}} - s_1 \right) \\ &\quad \times (\sqrt{1 - \theta^2 + \theta^2 s_1^2} - \theta s_1)^{n-4}. \end{aligned}$$

$$\begin{aligned}\frac{\partial g_1}{\partial \theta}(\rho, \theta, s) &= (n-3)(n-4)s_1^2(\rho - \theta s_1)^{n-5}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}} \\ &\quad - \alpha(n-3)s_1(1-s_1^2)\theta(\rho - \theta s_1)^{n-4}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-1}.\end{aligned}$$

$$\begin{aligned}\frac{\partial g_2}{\partial \theta}(\rho, \theta, s) &= -\alpha(n-3)s_1(1-s_1^2)\theta(\rho - \theta s_1)^{n-4}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-1} \\ &\quad + \alpha(\alpha-2)(1-s_1^2)^2\theta^2(\rho - \theta s_1)^{n-3}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-2} \\ &\quad + \alpha(1-s_1^2)(\rho - \theta s_1)^{n-3}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-1}.\end{aligned}$$

We set,

$$\begin{aligned}g_{11}(\rho, \theta, s) &= (n-3)(n-4)s_1^2(\rho - \theta s_1)^{n-5}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}}, \\ g_{12}(\rho, \theta, s) &= -2\alpha(n-3)s_1(1-s_1^2)\theta(\rho - \theta s_1)^{n-4}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-1}, \\ g_{21}(\rho, \theta, s) &= \alpha(\alpha-2)(1-s_1^2)^2\theta^2(\rho - \theta s_1)^{n-3}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-2}, \\ g_{22}(\rho, \theta, s) &= \alpha(1-s_1^2)(\rho - \theta s_1)^{n-3}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-1}.\end{aligned}$$

Let $a, b \in (0, 1/2)$ with $a < b$. We have, $\forall s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$,

$$\left| \frac{\partial g_1}{\partial \theta}(\rho, \theta, s) \right| \leq (n-3)(n-4)4^{n-5}a^\alpha(1-s_1^2)^{\frac{\alpha}{2}} + |\alpha|(n-3)4^{n-4}a^{\alpha-1}(1-s_1^2)^{\frac{\alpha}{2}}. \quad (1.7)$$

$$\begin{aligned}\left| \frac{\partial g_2}{\partial \theta}(\rho, \theta, s) \right| &\leq |\alpha(\alpha-2)|4^{n-3}a^{\alpha-2}(1-s_1^2)^{\frac{\alpha}{2}} \\ &\quad + |\alpha|4^{n-3}a^{\alpha-1}(1-s_1^2)^{\frac{\alpha}{2}} \\ &\quad + |\alpha|(n-3)4^{n-4}a^{\alpha-1}(1-s_1^2)^{\frac{\alpha}{2}}. \quad (1.8)\end{aligned}$$

Then, for any $i \in \{1, 2\}$, the maps $\theta \mapsto \int_0^z g_i(\rho, \theta, s)d\rho$ is differentiable on $(0, 1/2)$, and

$$\frac{\partial}{\partial \theta} \int_0^z g_i(\rho, \theta, s)d\rho = \int_0^z \frac{\partial g_i}{\partial \theta}(\rho, \theta, s)d\rho.$$

Furthermore, for any $i \in \{1, 2\}$, $\theta \mapsto \frac{\partial g_i}{\partial \theta}(\rho, \theta, s)$ is continuous on $(-1/2, 1/2) \setminus \{0\}$, then, $\theta \mapsto \int_0^z \frac{\partial g_i}{\partial \theta}(\rho, \theta, s)d\rho$, is continuous on $(-1/2, 1/2) \setminus \{0\}$. Hence, for any $i \in \{1, 2\}$ and for any $\epsilon > 0$, we have $\forall h, k$ two sufficiently small,

$$\begin{aligned}\left| \int_0^{z+h} \frac{\partial g_i}{\partial \theta}(\rho, \theta + k, s)d\rho - \int_0^z \frac{\partial g_i}{\partial \theta}(\rho, \theta, s)d\rho \right| &\leq \left| \int_0^z \frac{\partial g_i}{\partial \theta}(\rho, \theta + k, s)d\rho \right. \\ &\quad \left. - \int_0^z \frac{\partial g_i}{\partial \theta}(\rho, \theta, s)d\rho \right| \\ &\quad + \left| \int_z^{z+h} \frac{\partial g_i}{\partial \theta}(\rho, \theta + k, s)d\rho \right| \\ &\leq \epsilon.\end{aligned}$$

This proves that for any $i \in \{1, 2\}$, $(z, \theta) \mapsto \int_0^z \frac{\partial g_i}{\partial \theta}(\rho, \theta, s) d\rho$ is continuous on $[-1, 3] \times (-1/2, 1/2) \setminus \{0\}$. Moreover, for any $i \in \{1, 2\}$ the map $\rho \mapsto g_i(\rho, \theta, s)$ is continuous on $[-1, 3]$ for any $\theta \in (-1/2, 1/2) \setminus \{0\}$. Then, $z \mapsto \int_0^z g_i(\rho, \theta, s) d\rho$ is differentiable on $[-1, 3]$ for any $\theta \in (-1/2, 1/2) \setminus \{0\}$ and $\frac{\partial}{\partial z} \int_0^z g_i(\rho, \theta, s) d\rho = g_i(z, \theta, s)$. Since $(z, \theta) \mapsto g_i(z, \theta)$ is continuous on $[-1, 3] \times (-1/2, 1/2) \setminus \{0\}$ we finally deduce that for any $i \in \{1, 2\}$, $\theta \mapsto \int_{\theta s_1}^{\beta(\theta, s)} g_i(\rho, \theta, s) d\rho$ is differentiable on $(-1/2, 1/2) \setminus \{0\}$ and,

$$\begin{aligned} \sum_{i=1}^2 \frac{\partial}{\partial \theta} \int_{\theta s_1}^{\beta(\theta, s)} g_i(\rho, \theta, s) d\rho &= \frac{\theta(s_1^2 - 1)}{\sqrt{1 - \theta^2 + \theta^2 s_1^2}} \\ &\quad \times \left(-(n-3)s_1(\sqrt{1 - \theta^2 + \theta^2 s_1^2} - \theta s_1)^{n-4} \right. \\ &\quad \left. + \alpha\theta(1 - s_1^2)(\sqrt{1 - \theta^2 + \theta^2 s_1^2} - \theta s_1)^{n-3} \right) \\ &\quad + \sum_{i=1}^2 \int_{\theta s_1}^{\beta(\theta, s)} \frac{\partial^2 g_i}{\partial^2 \theta}(\rho, \theta, s) d\rho. \end{aligned}$$

We deduce that $\theta \mapsto \frac{\partial F}{\partial \theta}$ is differentiable in $(-1/2, 1/2) \setminus \{0\}$. Moreover, we see that the map,

$$\begin{aligned} \theta \mapsto \lambda(\theta, s) &= \frac{\theta(s_1^2 - 1)}{\sqrt{1 - \theta^2 + \theta^2 s_1^2}} \left(-(n-3)s_1(\sqrt{1 - \theta^2 + \theta^2 s_1^2} - \theta s_1)^{n-4} \right. \\ &\quad \left. + \alpha\theta(1 - s_1^2)(\sqrt{1 - \theta^2 + \theta^2 s_1^2} - \theta s_1)^{n-3} \right) \end{aligned}$$

is indefinitely differentiable on $(-1/2, 1/2) \times \mathbb{S}^{n-1}$. Then, by (1.1), (1.2), (1.8), (1.7), (1.3) and (A), for any $a, b \in (0, 1/2)$, $a < b$ there exists constants $K_{1,n,ab,\alpha}$, $K_{2,n,ab,\alpha}$, $K_{3,n,ab,\alpha}$ so that, for any $|\theta| \in (a, b)$, for any $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$,

$$\left| \frac{\partial^2 HF}{\partial \theta^2}(\theta, s) \right| \leq K_{1,n,ab,\alpha}(1 - s_1^2)^{\frac{\alpha}{2}} + K_{2,n,ab,\alpha}(1 - s_1^2)^{\frac{3-n}{2}} + K_{3,n,ab,\alpha}.$$

We deduce by Lebesgue Theorem that the map $\theta \mapsto E(\theta)$ is two times differentiable on $(-1/2, 1/2) \setminus \{0\}$ and,

$$\frac{d^2 G}{d\theta^2}(\theta) = \int_{\mathbb{S}^{n-1}} \frac{\partial^2 HF}{\partial \theta^2}(\theta, s) ds.$$

Lemma 0.7 *If $5 - n > \alpha > 4 - n$, the map $\theta \mapsto G(\theta)$ is two times differentiable at 0.*

Proof Suppose that $\alpha \in (4 - n, 5 - n)$. As in Lemma 1.5, we can see that,

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \frac{\partial^2 H}{\partial \theta^2}(\theta, s) F(\theta, s) ds &\xrightarrow{\theta \rightarrow 0} \int_{\mathbb{S}^{n-1}} \frac{1}{2} \frac{(2n-3)s_1^2 - (n-2)}{n-2+\alpha} ds = \frac{-n^2 + 4n - 3}{2n(n-2+\alpha)} |\mathbb{S}^{n-1}|, \\ \int_{\mathbb{S}^{n-1}} \frac{\partial H}{\partial \theta}(\theta, s) \eta(\theta, s) ds &\xrightarrow{\theta \rightarrow 0} 0, \\ \int_{\mathbb{S}^{n-1}} \frac{\partial H}{\partial \theta}(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho &\xrightarrow{\theta \rightarrow 0} \int_{\mathbb{S}^{n-1}} \frac{2(n-3)(n-1)}{n-3+\alpha} s_1^2 = \frac{2(n-3)(n-1)}{n(n-3+\alpha)} |\mathbb{S}^{n-1}|, \\ \int_{\mathbb{S}^{n-1}} \frac{\partial H}{\partial \theta}(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho &\xrightarrow{\theta \rightarrow 0} 0, \\ \int_{\mathbb{S}^{n-1}} H(\theta, s) \frac{\partial \eta}{\partial \theta}(\theta, s) ds &\xrightarrow{\theta \rightarrow 0} \int_{\mathbb{S}^{n-1}} (n-1)(s_1^2 - 1) ds = \frac{-(n-1)^2}{n} |\mathbb{S}^{n-1}|, \end{aligned}$$

and

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \lambda(\theta, s) ds \xrightarrow{\theta \rightarrow 0} 0.$$

As in Lemma 1.5, we set $\rho = \sqrt{1 - s_1^2} \theta t$ if $\theta > 0$. Hence,

$$\int_{\theta s_1}^{\beta(\theta, s)} g_{11}(\rho, \theta, s) d\rho = (n-3)(n-4)s_1^2(1-s_1^2)^{\frac{1+\alpha}{2}} \theta^{n-4+\alpha} J(n-5, \frac{\alpha}{2}).$$

$$\int_{\theta s_1}^{\beta(\theta, s)} g_{12}(\rho, \theta, s) d\rho = -2\alpha(n-3)s_1(1-s_1^2)^{\frac{1+\alpha}{2}} \theta^{n-4+\alpha} J(n-4, \frac{\alpha}{2} - 1)$$

$$\int_{\theta s_1}^{\beta(\theta, s)} g_{21}(\rho, \theta, s) d\rho = \alpha(\alpha-2)(1-s_1^2)^{\frac{1+\alpha}{2}} \theta^{n-4+\alpha} J(n-3, \frac{\alpha}{2} - 2)$$

$$\int_{\theta s_1}^{\beta(\theta, s)} g_{22}(\rho, \theta, s) d\rho = \alpha(1-s_1^2)^{\frac{1+\alpha}{2}} \theta^{n-4+\alpha} J(n-3, \frac{\alpha}{2} - 1)$$

Since $\alpha \in (4 - n, 5 - n)$, the integrals $J(n-5, \frac{\alpha}{2})$ and $J(n-3, \frac{\alpha}{2} - 1)$ are infinite and we have,

$$J(n-5, \frac{\alpha}{2}) \underset{0}{\sim} \frac{(1-s_1^2)^{\frac{-1-\alpha}{2}} \theta^{-n-\alpha+4}}{n-4+\alpha}, \quad J(n-3, \frac{\alpha}{2} - 1) \underset{0}{\sim} \frac{(1-s_1^2)^{\frac{1-\alpha}{2}} \theta^{-n-\alpha+4}}{n-4+\alpha}$$

And the integrals

$J(n-4, \frac{\alpha}{2}-1)$ and $J(n-3, \frac{\alpha}{2}-2)$ are finite. Then,

$$\int_{\theta_{s_1}}^{\beta(\theta,s)} g_{11}(\rho, \theta, s) d\rho \xrightarrow{\theta \rightarrow 0} \frac{(n-3)(n-4)s_1^2}{n-4+\alpha}, \quad \int_{\theta_{s_1}}^{\beta(\theta,s)} g_{22}(\rho, \theta, s) d\rho \xrightarrow{\theta \rightarrow 0} \frac{\alpha(1-s_1^2)}{n-4+\alpha}.$$

$$\int_{\theta_{s_1}}^{\beta(\theta,s)} g_{12}(\rho, \theta, s) d\rho \xrightarrow{\theta \rightarrow 0} 0, \quad \int_{\theta_{s_1}}^{\beta(\theta,s)} g_{21}(\rho, \theta, s) d\rho \xrightarrow{\theta \rightarrow 0} 0$$

Moreover, we can see that, for any $i, j \in \{1, 2\}$, for any $(\theta, s) \in (-1/2, 1/2) \times \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$,

$$H(\theta, s) \int_{\theta_{s_1}}^{\beta(\theta,s)} g_{ij}(\rho, \theta, s) d\rho \leq C_{n,\alpha}(1-s_1^2)^{\frac{5-n}{2}} + D_{n,\alpha}(1-s_1^2)^{\frac{\alpha+1}{2}}.$$

where $C_{n,\alpha}$ and $D_{n,\alpha}$ are two constants independent of θ . By Lebesgue Theorem we deduce that,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \frac{\partial^2 F}{\partial \theta^2}(\theta, s) ds \xrightarrow{\theta \rightarrow 0} \frac{-(n-1)^2}{n} |\mathbb{S}^{n-1}|$$

$$+ (n-1) \frac{(n-3)(n-4) + \alpha(n-1)}{n(n-4+\alpha)} |\mathbb{S}^{n-1}|.$$

By Lemmas 1.1, 1.2, 1.3, $\theta \mapsto G(\theta) \in \mathcal{C}^1((-1/2, 1/2), \mathbb{R})$ and is two times differentiable on $(-1/2, 1/2) \setminus \{0\}$. Furthermore, when $\alpha \in (4-n, 5-n)$, as the limit of $\frac{d^2 G}{d\theta^2}(\theta)$ exists as $\theta \rightarrow 0$, we have $\theta \mapsto G(\theta)$ is two times differentiable on $(-1/2, 1/2)$.

Proof of ii). Assume that $\alpha \in (4-n, 5-n)$, by Lemma 1.1, 1.2, 1.3, 1.4, we have,

$$G(\theta) = G(0) + \frac{1}{2} \frac{d^2 G}{d\theta^2}(0) + o(\theta^2).$$

Furthermore we have,

$$\frac{d^2 G}{d\theta^2}(0) = \frac{-n^2 + 4n - 3}{2n(n-2+\alpha)} |\mathbb{S}^{n-1}| + \frac{2(n-3)(n-1)}{n(n-3+\alpha)} |\mathbb{S}^{n-1}|$$

$$+ \frac{-(n-1)^2}{n} |\mathbb{S}^{n-1}| + (n-1) \frac{(n-3)(n-4) + \alpha(n-1)}{n(n-4+\alpha)} |\mathbb{S}^{n-1}|.$$

We have, for any $n \geq 6$.

$$(n-3)(n-4) + \alpha(n-1) \xrightarrow{\alpha \rightarrow 4-n} -2(n-4) < 0.$$

Then,

$$\frac{(n-3)(n-4) + \alpha(n-1)}{n(n-4+\alpha)} \xrightarrow{\alpha \rightarrow > 4-n} -\infty, \text{ and } \frac{d^2 G}{d^2 \theta}(0) \xrightarrow{\alpha \rightarrow > 4-n} -\infty.$$

Hence, there is α_0 such that, for any $\alpha \in (4 - n, \alpha_0)$, $G(\theta) < G(0)$ for θ sufficiently small, that is,

$$G(\theta) = E_{2,r^\alpha}(u_a) = \int_{\mathbf{B}^n} r^\alpha \|\nabla u_a\|^2 dx < G(0) = \int_{\mathbf{B}^n} r^\alpha \|\nabla u_0\|^2 dx. \quad \blacksquare$$

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